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The Existence of a Discontinuous Homomorphism Requires a Strong Axiom of Choice

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ABSTRACT

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Conner and Spencer used ultrafilters to construct homomorphisms between fundamental groups that could not be induced by continuous functions between the underlying spaces. We use methods from Shelah and Pawlikowski to prove that Conner and Spencer could not have constructed these homomorphisms with a weak version of the Axiom of Choice. This led us to define and examine a class of pathological objects that cannot be constructed without a strong version of the Axiom of Choice, which we call the class of inscrutable objects. Objects that do not need a strong version of the Axiom of Choice are scrutable. We show that the scrutable homomorphisms from the fundamental group of a Peano continuum are exactly the homomorphisms induced by a continuous function.

We suspect that any proposed theorem whose proof does not use a strong Axiom of Choice cannot have an inscrutable counterexample.

Keywords: inscrutable, inscrutability, scrutable, scrutability, axiom of choice, discontinuous, locally trivial, non-locally trivial, kernel invariance, shelah, pawlikowski, countable choice, choice, arbitrary choice, dependent choice, discontinuity

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I would like to thank my parents, who love and support me. They are my best examples, and I love them dearly.

Michael Andersen

Contents

Chapter 1. Introduction

Conner and Spencer use ultrafilters to construct non-trivial homomorphisms from the Hawaiian earring group to G, a finite group, such that every group element corresponding to a simple closed curve is mapped trivially. It is not difficult to see that such maps cannot be induced by continuous maps of the underlying spaces.

We use methods by Shelah and Pawlikowski to show that every homomorphism from the fundamental group of a Peano continuum to a countable group which can be constructed using the Axiom of Dependent Choice is induced by a continuous map of the underlying spaces.

This inspired us to define a class of inscrutable objects, those that can not be constructed using only the axiom of Dependent Choice. Our theorem can then be restated that the homomorphisms from the fundamental group of a Peano continuum to a countable group that can not be induced by a continuous function of the underlying spaces are exactly the inscrutable homomorphisms between those groups.

Definition 2.1 (Loop based at x)**.** A *loop based at x* is a continuous function $f\colon I \to X$ such that $f(0) = f(1) = x$.

Definition 2.2 (Freely homotopic)**.** Let X and Y be a topological spaces and $f, g\colon X \to Y$ be continuous functions. f and g are *freely homotopic* if there exists a continuous function $\phi\colon [0,1] \times X \to Y$ such that $\phi\mid_{0\times X} = f$ and $\phi\mid_{1\times X} = g$. ϕ is called a *free homotopy.*

Furthermore, if f and g are loops based at x and $\phi\mid_{I\times 0,1}$ is constant, we say that f and g are *homotopic.* Homotopy is an equivalence relation and is denoted $[\alpha]$.

Definition 2.3 (Fundamental group of a space X at x)**.** The *fundamental group of a space X at x*, denoted $\pi_1(X, x)$, is a group whose underlying set is the set of homotopy classes of loops of X based at x, and whose binary operation is $[\alpha][\beta] = [\alpha * \beta]$, where $*$ is concatenation. It is easy to check that the set of homotopy equivalence classes of X with this operation is a group.

Due to the underlying set of $\pi_1(X, x)$ being a set of classes of loops based at x we will only consider topological spaces that are path connected.

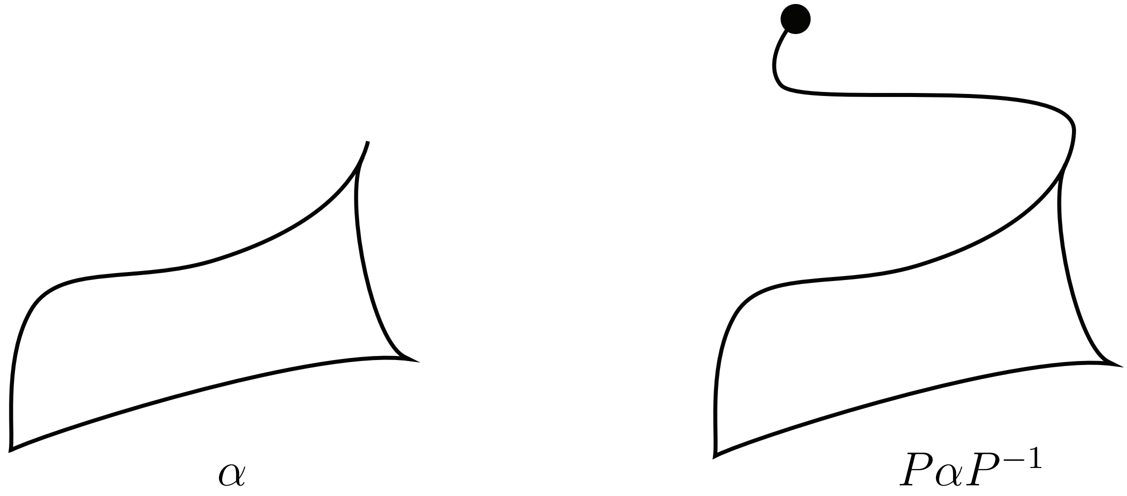
Lemma 1. *The fundamental group of a space X does not depend on the choice of x, up to isomorphism.*

Proof. Let $x_0, x_1 \in X$ and P be a path from x_1 to x_0. There exists an isomorphism, called the P induced isomorphism, $i_P\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$, given by $i_P([\alpha]) = [P * \alpha * P^{-1}]$. So $\pi_1(X)$ does not depend on the choice of basepoint. □

We note that there is no cannonical isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$, as distinct choices of P may give distinct isomorphisms.

We say that $\phi\colon \pi_1(X, x_0) \to G$ induces $\phi'\colon \pi_1(X, x_1) \to G$ if for some path P, $\phi = \phi' \circ i_P$. We note that if ϕ induces ϕ', then ϕ' induces ϕ by $\phi' = \phi' \circ (i_P \circ i_{P^{-1}}) = \phi \circ i_{P^{-1}}$.

Figure 2.1: Conjugation by a path P



Definition 2.4 (Trivial free homotopy class relative to ϕ)**.** Let H be a free homotopy class of loops of X. H is *trivial relative to* $\phi\colon \pi_1(X,x) \to G$ if there exists a loop $\beta \in H$ based at x such that $[\beta] \in \ker(\phi)$. We denote this property by $H \in \ker\phi$.

Lemma 2. *A trivial free homotopy class relative to* $\phi\colon \pi_1(X,x_0) \to G$ *is trivial relative to* $\phi'\colon \pi_1(X,x_1) \to G$*, if* ϕ' *is induced by* ϕ*.*

Proof. Let $H \in \ker(\phi\colon \pi_1(X,x_0) \to G)$. There exists $\alpha \in H$ such that $[\alpha] \in \ker\phi$. Let P be a path such that $\phi = \phi' \circ i_P$. Then $P\alpha P^{-1} \in H$ and $\phi'([P\alpha P^{-1}]) = \phi'(i_P([\alpha])) = \phi([\alpha])$, which is trivial. So $H \in \ker\phi'$

□

Lemmas 1 and 2 allow us to omit referencing the basepoint of the fundamental group of X because the kernel of a homomorphism is invariant. We will use $\pi_1(X)$ to denote the fundamental group and not mention a basepoint.

Definition 2.5 (A sequence of loops converging to p)**.** A *sequence of loops* $\{\alpha_i\}$ *converges to* p if for every open set U containing p there exists N such that $\alpha_i \subset U$ for all $i > N$.

For locally path connected spaces, there exists a path in U for each $i > N$ connecting p to the basepoint of α_i.

Lemma 3. *Let Z_i be compact, $g_i \colon Z_i \to X$ be continuous functions, $x \in X$, $p_i = g_i^{-1}(x)$, and W be the one point compactification of $\bigsqcup(Z_i \setminus \{p_i\})$. Then $\bigsqcup g_i \colon W \to X$ is a continuous function if the basepoint of W is mapped to $g_i(p_i)$.*

Proof. Let $U \subset X$ be an open set. The inverse image of U under each g_i is open in its respective Z_i. A set containing the basepoint of W is open in W if and only if it is the union over all i of non-empty open sets V_i such that $p_i \in V_i \subset Z_i$, so $\bigsqcup g_i^{-1}(U)$ is open in W if $x \in U$. A set not containing the basepoint is open if and only if it is a union of open sets $V_i \subset Z_i$, so $\bigsqcup g_i^{-1}(U)$ is open in W if $x \notin U$.

□

Definition 2.6 (Hawaiian earring)**.** The *Hawaiian earring*, denoted *HE*, is the one point compactification of a sequence of copies of $S^1 \setminus \{p\}$. The point of compactification is called the basepoint of the Hawaiian earring. We use c_i to represent the i^{th} copy of S^1 in the sequence. The homotopy class of c_i is denoted $l_i = [c_i]$. We will let f_i denote some fixed parameterization of c_i. The fundamental group of the Hawaian earring is called the Hawaiian earring group, and we denote it *HEG*.

Definition 2.7 (Cantor set)**.** We use $C = \{0,1\}^{\mathbb{N}}$ endowed with the product topology as our model of the *Cantor set*. Every element of the Cantor set is represented by a sequence of 1's and 0's, and the i^{th} term of the representation is called the i^{th} letter of that element. We use e_i to represent the element of the Cantor set that has a 1 in the i^{th} position and 0's elsewhere. The *parity* of the i^{th} letter is the choice of 0 or 1.

Definition 2.8 (Polish space)**.** A *Polish space* is a metric space that is complete and separable.

Definition 2.9 (Hausdorff space)**.** A *Hausdorff space* is a topological space X such that every pair of points $x, y \in X$ implies the existence of disjoint open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.10 (Continuum)**.** A *continuum* is a non-empty, compact, connected, metric space. A Peano continuum is a locally connected continuum. It is known that Peano continua are locally path connected.

Definition 2.11 (ZF)**.** We use ZF to denote the Zermelo-Fraenkel axioms of set theory.

Definition 2.12 (Axiom of Dependent Choice)**.** The Axiom of Dependent Choice, denoted DC, states: For any nonempty set X and any entire binary relation $\sim$ on X, there is a sequence (x_n) in X such that $x_n \sim x_{n+1}$ for each $n \in \mathbb{N}$.

Definition 3.1 (Baire space)**.** A topological space X is a *Baire space* if $\text{int}(\cup A_n)$ is empty for every countable collection $\{A_n\}$ of closed sets with empty interiors.

The following is the Baire category theorem, as stated in Munkres [3, Thm 48.2, P. 296].

Theorem 3.2. *If X is either a compact Hausdorff space or a complete metric space, then X is a Baire space.*

Definition 3.3 (Nowhere dense in X)**.** A set $A \subset X$ is *nowhere dense* in X if every non-empty open set intersecting A contains a non-empty open set that does not intersect A. We will omit X if the ambient space is obvious.

Definition 3.4 (Meager set in X)**.** A set of X is *meager in X* if it is the countable union of sets that are nowhere dense in X. We will omit X if the ambient space is obvious.

Lemma 4. *Let X be a topological space*

(i) The countable union of meager sets of is a meager set.

(ii) A nowhere dense set has empty interior.

(iii) A closed set with empty interior is nowhere dense.

(iv) The closure of a nowhere dense set is nowhere dense.

(v) Countable subsets of Hausdorff spaces are meager.

(vi) The subset of a meager set is meager.

(vii) If $f\colon X \to Y$ is a quotient map and $M \subset Y$ is a nowhere dense set, then $f^{-1}(M)$ is a nowhere dense set.

(viii) If $f\colon X \to Y$ is a quotient map and $M \subset Y$ is a meager set, then $f^{-1}(M)$ is a meager set.

Proof. (i) The countable union of a countable union of nowhere dense sets is the countable union of nowhere dense sets.

(ii) Let A be nowhere dense. Every non-empty open subset of $\text{int}(A)$ is a subset of A, and every open set that intersects A contains a non-empty open set that does not intersect A. So there cannot be a non-empty open subset of $\text{int}(A)$.

(iii) Let A be closed in X and have empty interior as a set of X. Since no point of A is in the interior of A, every open set U containing a point of A contains a point not in A, call it p. The complement of A is open, so there exists an open subset of U containing p that does not intersect A.

(iv) Let A be a nowhere dense set and let $\overline{A}$ be the closure of A. Since A has empty interior, every point of $\overline{A}$ is either an element of the boundary of A or is a limit point of the boundary of A, so every open set containing a point of $\overline{A}$ contains a point of the complement of $\overline{A}$, therefore $\overline{A}$ has empty interior implying that $\overline{A}$ is nowhere dense.

(v) If X is a Hausdorff space then singleton set of X is a closed set with empty interior, so every countable union of points of X is a meager set.

(vi) It is obvious that a subset of a nowhere dense set is nowhere dense, and that the subset of a meager set is a countable union of subsets of nowhere dense sets.

(vii) If A is an open set of X, then $f(A)$ is an open set of Y, so $f(A)$ contains an open set B that does not intersect M, so $A \cap f^{-1}(B)$ is an open subset of A that does not intersect $f^{-1}(M)$. So $f^{-1}(M)$ is nowhere dense.

(viii) If $A \subset Y$ is a meager set, it is the countable union of nowhere dense sets. Each of these nowhere dense sets has a nowhere dense pre-image under f, so $f^{-1}(A)$ is the countable union of nowhere dense sets.

□

Lemma 5. *A non-empty Baire space is not meager in itself.*

Proof. Let X be a Baire space. Suppose that $\{A_n\}$ is a countable collection of nowhere dense sets such that $\cup A_n = X$. For each n, $\overline{A}_n$ is a nowhere dense set, so $\overline{A}_n$ is a closed set with empty interior (lemma 4), so $\text{int}(X) = \text{int}\big(\cup \overline{A_n}\big) = \emptyset$, contradicting $\text{int}(X) = X$. □

Definition 3.5 (Symmetric difference)**.** The *symmetric difference* is a binary operation on the subsets of a space, and is denoted $\triangle$. If $A, B \subset X$, then $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Lemma 6. *Let $f\colon X \to Y$ be a quotient map, $U, M \subset Y$, and let*

$$B = \big(f^{-1}(U) \triangle f^{-1}(M)\big) \setminus f^{-1}(U \triangle M),$$

then

(i) $f^{-1}(U \triangle M) \subset f^{-1}(U) \triangle f^{-1}(M)$

(ii) $B \subset f^{-1}(M)$

(iii) $f^{-1}(U \triangle M) = f^{-1}(U) \triangle \big(f^{-1}(M) \setminus B\big)$

Proof. (i) If $x \in f^{-1}(U \triangle M)$, then $f(x) \in U$ or $f(x) \in M$ but not both, so $x \in f^{-1}(U)$ or $f^{-1}(M)$ but not both, so $x \in f^{-1}(U) \triangle f^{-1}(M)$.

We notice that the reverse containment does not necessarily hold. If $x \in U \setminus M$ and $y \in M \setminus U$ map to the same point of Y, then $x, y \in f^{-1}(U) \triangle f^{-1}(M)$, but $x, y \notin f^{-1}(U \triangle M)$.

(ii) If $x \in B$, either $f(x) \in U$ or $f(x) \in M$. If $f(x) \notin U \cap M$, then $f(x) \in U \triangle M$, so $x \in f^{-1}(U \triangle M)$, so $f(x) \in U \cap M$, which is a contradiction, so $x \in f^{-1}(M)$.

(iii) B is a subset of $f^{-1}(M)$ and of $f^{-1}(U)$ by *(ii)*, so deleting B from M would include B in the symmetric difference of the resulting sets.

□

Definition 3.6 (Property of Baire)**.** A subset A of a topological space has the *property of Baire* if $A = U \triangle M$, for some open set U and some meager set M.

Lemma 7. *If $f\colon X \to Y$ is a quotient map and $A \subset Y$ has the property of Baire, then $f^{-1}(A)$ has the property of Baire.*

Proof. Let U be open and M be meager such that $A = U \triangle M$. Let B be as in lemma 6. By lemma 4, $f^{-1}(M)$ is meager, so $f^{-1}(M) \setminus B$ is meager, so

$$f^{-1}(U \triangle M) = f^{-1}(U) \triangle \big(f^{-1}(M) \setminus B\big)$$

has the property of Baire.

□

Definition 4.1 (Locally trivial relative to ϕ)**.** Let X be a topological space and $\phi\colon \pi_1(X) \to G$ be a homomorphism. X is *locally trivial relative to* ϕ if for every point $p \in X$ there exists a neighborhood U containing p such that the free homotopy class of every loop $\alpha \subset U$ is in $\ker\phi$. We will denote "X is locally trivial relative to ϕ" by "X is locally trivial (rel ϕ)". Note that the choice of basepoint does not matter, because $\ker\phi$ is a normal subgroup of $\pi_1(X)$.

Definition 4.2 (Two set simple cover relative to ϕ)**.** A *two set simple cover* (rel ϕ) is a cover of a topological space X such that for all $[\gamma] \in \pi_1(X)$ such that whenever γ is contained in the union of two elements of the cover, $[\gamma] \in \ker(\phi)$. Note that this is well defined, since $\ker(\phi)$ is independent of basepoint in X. We will denote "$\mathcal{C}$ is a two set simple cover relative to ϕ" by "$\mathcal{C}$ is a two set simple cover rel(ϕ)".

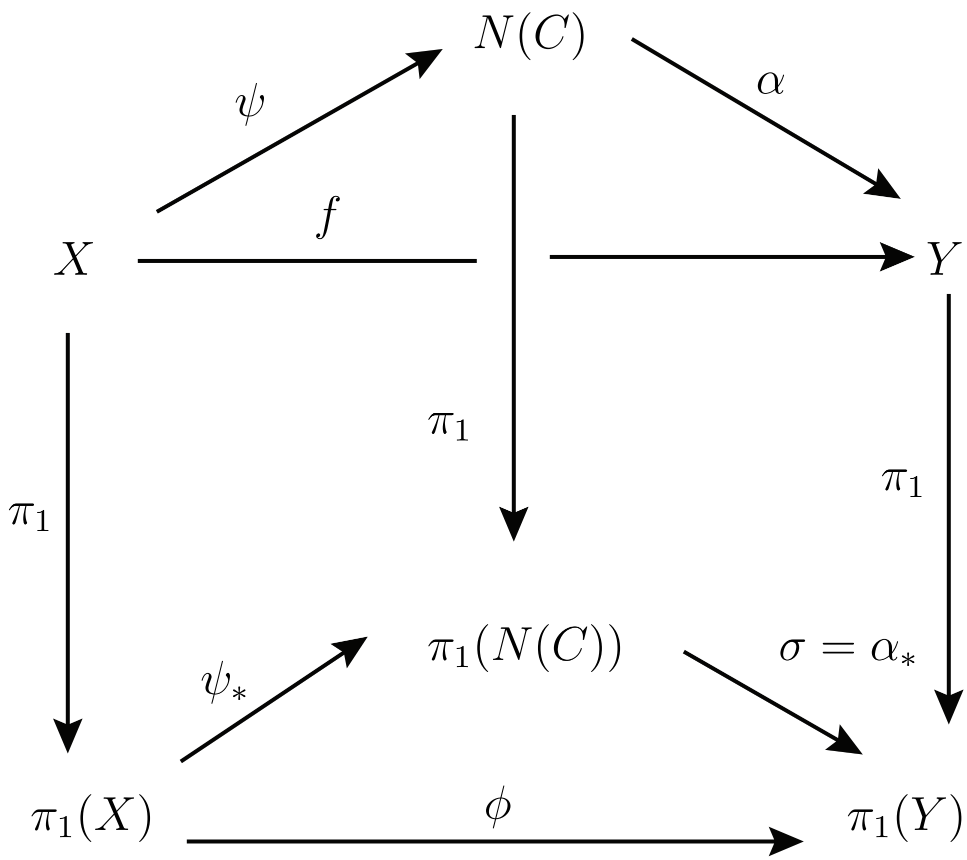
Lemma 8. *Let X be a topological space and Y a locally simply connected space. If $f\colon X \to Y$ is continuous then X is locally trivial (rel f_*)*

Proof. Let X be a topological space, Y a locally simply connected space, and $f\colon X \to Y$ be continuous. Let U be a simply connected open set about $f(p)$. Then any loop based at p and contained in $f^{-1}(U)$ lies in $\ker f_*$. So X is locally trivial (rel f_*). □

Lemma 9. *Let X be a Peano continuum and Y an aspherical simplicial complex. If $\phi\colon \pi_1(X) \to \pi_1(Y)$ has the property that X is locally trivial (rel ϕ) then there exists a continuous function $f\colon X \to Y$ such that f induces ϕ.*

Proof. Let X be a Peano continuum, Y be an aspherical simplicial complex, and $\phi\colon \pi_1(X) \to \pi_1(Y)$ have the property that X is locally trivial (rel ϕ). Since X is locally trivial (rel ϕ) we may cover X with open sets that are trivial with respect to ϕ. Let l be the Lesbesgue number of this cover. Cover X by balls of radius $l/3$, Since X is locally path connected, components of elements of this cover are path connected. There exists a finite subcover of

Figure 4.1: Commutative Diagram for Lemma 9



this cover, call it C. Since the union of intersecting elements of C has radius at most $2l/3$, C is two set simple (rel ϕ). Let $N(C)$ denote the nerve of C.

By [3] there exist homomorphisms $\psi_*\colon \pi_1(X) \to \pi_1(N(C))$ and $\sigma\colon \pi_1(N(C)) \to \pi_1(Y)$ such that $\phi = \sigma \circ \psi_*$, and ψ_* is induced by a continuous function $\psi\colon X \to N(C)$. This reduces the problem of finding a continuous function $f\colon X \to Y$ that induces ϕ to finding a function $h\colon N(C) \to Y$ that induces σ.

Since $N(C)$ is a simplicial complex and Y is an aspherical simplicial complex, there exists $\alpha\colon N(C) \to Y$ such that $\alpha_* = \sigma$. $f = \alpha \circ \psi$ induces $\sigma \circ \psi_* = \phi$ and the lemma is proved.

□

Lemma 10. *Let X be a first countable space and $\phi\colon \pi_1(X) \to G$. X is locally trivial (rel ϕ) if and only if for every continuous function $g\colon \text{HE} \to X$, $\phi([g(c_i)])$ is trivial for some i.*

Proof. ($\Rightarrow$) Suppose there exists a continuous function $g\colon HE \to X$ such that $\phi([g(c_i)])$ is non-trivial for all i. By continuity, $g(c_i)$ converges to a point, so every open set containing the point of convergence contains a loop mapped non-trivially by ϕ.

($\Leftarrow$) Let X be first countable and not locally trivial (rel ϕ). There exists $p \in X$ such that every neighborhood of p contains a loop α such that $\phi([\alpha])$ is not trivial. X is first countable, so there exists a sequence of nested open sets $\{U_i\}$ whose intersection is $\{p\}$. For each U_i choose a loop $\alpha_i \subset U_i$ based at p such that $\phi([\alpha_i])$ is not trivial. We now have a sequence of loops based at p, such that no loop of the sequence is mapped trivially.

We construct $g\colon HE \to X$ by mapping c_i of HE to α_i, with the basepoint of HE being mapped to $p \in X$. This gives a continuous function from HE to X such that $\phi(g(c_i)) = \phi(\alpha_i) \neq 1$ for all i.

□

Definition 5.1 (The Shelah function)**.** We define *the Shelah function* $h\colon C \to HEG$ by

$$a \mapsto [\alpha].$$

Where α is the loop given by

$$\alpha(x) = \begin{cases} f_i\big(2^i(x - 1 + \frac{1}{2^i})\big) & x \in \big[1 - \frac{1}{2}^i, 1 - \frac{1}{2}^{i+1}\big] \text{ and } a_i = 1 \\ x_0 & x \in \big[1 - \frac{1}{2}^i, 1 - \frac{1}{2}^{i+1}\big] \text{ and } a_i = 0. \end{cases}$$

Recall that f_i is a parameterization of c_i.

Definition 5.2 (The induced relation of ϕ)**.** Let $\phi\colon HEG\to G$ be a homomorphism. The *induced relation of* ϕ, denoted $\sim_\phi$, is given by $a \sim_\phi b \Leftrightarrow \phi(h(a)) = \phi(h(b))$, where a and b are elements of C. Note that $a \sim_\phi b \Leftrightarrow (h(a))[h(b)]^{-1} \in \ker(\phi)$.

Definition 5.3 (Property of Shelah)**.** An equivalence relation $\sim$ on C has the *property of Shelah* if for every pair $a, b \in C$, $\big|\{i|a_i \neq b_i\}\big| = 1 \Rightarrow a \nsim b$.

Lemma 11. *If ϕ: HEG$\to G$ maps all l_i's non-trivially then the induced relation of ϕ has the property of Shelah.*

Proof. Suppose that $a, b \in C$ such that $\{i \mid a_i \neq b_i\} = \{j\}$ With no loss of generality $h(a) = ul_jv$ and $h(b) = uv$. Then

$$\phi\big(h(a)(h(b))^{-1}\big) = \phi\big((ul_jv)(v^{-1}u^{-1})\big) = \phi(ul_ju^{-1}) = \phi(u)\phi(l_j)\big(\phi(u)\big)^{-1}$$

which is conjugate to a non-trivial element. So $a \nsim b$. □

Lemma 12. *[4, p. 3086 Lemma 4] A relation on the Cantor set with the property of Shelah and the property of Baire is meager.*

Pawlikowski does not use a version of the Axiom of Choice stronger than the Countable Axiom of Choice in the proof of this lemma.

Lemma 13. *If* ϕ: HEG$\to G$ *maps all* l_i*'s non-trivially, then the induced relation of* ϕ *does not have the property of Baire.*

Proof. Assume ϕ: *HEG*$\to G$ maps all l_i's non-trivially and $\sim_\phi$ has the property of Baire. By lemma 11 it also has the property of Shelah, so $\sim_\phi$ is a meager set in $C \times C$ by lemma 12.

If $[a]$ is the equivalence class of $a \in C$ under $\sim_\phi$, then $[a] \times [a] \subset \sim_\phi$ must be meager in $C \times C$, so $[a]$ is meager in C by the Kuratowski-Ulam theorem [6, Thm 3.5.16, P. 112]. The union of the equivalence classes of $\sim_\phi$ is C, so C is the countable union of meager sets, which implies that C is meager by lemma 4, which contradicts lemma 5.

□

Definition 5.4 (Inscrutable property)**.** A property P is *inscrutable* if

ZFC implies the existence of an object with property P

and

ZF + DC + "P is false" is equiconsistent with ZFC.

Definition 5.5 (Continuity)**.** A homomorphism ϕ from the fundamental group of a Peano continuum to a countable group is *continuous* if there exists a function f from a Peano continuum to an aspherical symplicial complex such that $f_* = \phi$. A homomorphism ϕ from the fundamental group of a Peano continuum to a countable group is *discontinuous* otherwise.

Theorem 5.6. *ZF + DC + "no homomorphism is discontinuous" is equiconsistent with ZFC.*

Proof. (⇒) Assume that ZF + DC + "no homomorphism is discontinuous" is consistent. Gödel [2, P. 53] showed that the consistency of ZF implies the consistency of ZFC.

(⇐) Let X be a Peano continuum and assume that ZFC is consistent. Shelah showed that ZF + DC + "Every subset of $\mathbb{R}$ has the Baire property" is consistent is implied by the consistency of ZFC [5, p. 43].

Suppose $\phi\colon \pi_1(X) \to G$ is a discontinuous homomorphism from the fundamental group of a Peano continuum to a countable group. Lemma 9 implies that X is not locally trivial (rel ϕ) and Lemma 10 implies that there exists a continuous $g\colon HE \to X$ such that $\phi([g(c_i)])$ is nontrivial for all i.

Lemma 13 implies that $\sim_{\phi\circ g_*}$ is a subset of $C \times C$ that does not have the property of Baire. Since $C \times C$ is homeomorphic to C, and $\mathbb{R}$ is a quotient of C, lemma 7 implies we have constructed a subset of $\mathbb{R}$ that does not have the property of Baire.

This implies that the existence of a discontinuous homomorphism contradicts ZF + DC + "Every subset of $\mathbb{R}$ has the property of Baire." So ZF + DC + "no homomorphism is discontinuous" is consistent.

□

Theorem 5.7. *Discontinuity is an inscrutable property.*

Proof. Conner and Spencer [1, p. 225] use ZFC to construct a homomorphism g from HEG to a finite group such that g cannot be induced by a homomorphism of the underlying spaces. So ZF + DC + "No homomorphism is discontinuous " is equiconsistent with ZFC, by Theorem 5.6.

□

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